Inverse Transform of Duhamel Integral for Data Processing in Hydrology

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Abstract

The inverse transform of Duhamel integral is a procedure to find the value of one of the integrands of a Duhamel integral knowing the other and the result of the integral. It corresponds to the determination of the function of influence or the inverse transform of the output to the input for a system. This paper compares the methods of the inverse transform which have been proposed as the general procedures for data processing in hydrology and the related fields. It is shown that the method of Kuchment gives a much better result than the least-square method, and that the Fourier transform method can give the best result when the optimal filter is designed according to the individual problem. In the example in which the method of Kuchment is applied to the analysis of data from the tracing of the overland flow in a model drainage basin, the mean velocity and the coefficient of dispersion of the tracer are evaluated to illustrate the effectiveness of this method as a general procedure for a detailed and quantitative analysis of hydrologic processes.

1. Introduction

The relation between the input and output time series of any linear system can be described by a functional called "convolution" or "Duhamel integral" as follows,

$$g(t)=\int_{-\infty}^{t}\phi(t-\tau)f(\tau)d\tau, \tag{1}$$

where t is the time, $f(t)$ the input, $g(t)$ the output, and $\phi(t)$ the function of influence or the kernel function. The inverse transform of Duhamel integral is to solve (1) as an integral equation. It involves two kinds of transform, (A) to calculate $\phi(t)$ knowing $f(t)$ and $g(t)$, and (B) to calculate $f(t)$ knowing $\phi(t)$ and $g(t)$. The former problem corresponds to analyzing the characteristics of a system. This can be directly done from the response of the system against an inpulse input (e.g. unit graph method). For hydrologic systems, however, it is often impracticable because of difficulty in controlling the input so that it becomes an inpulse input. The second problem occurs frequently in the field of geoscience*. The most of the natural phenomena that occur on and in the earth are unobservable. Therefore, when the output of a geoscientific system is observed, it is frequent that the input is unobservable.

An essential difficulty is imposed on the problem of the kind of (B) in the field of hydrology. Except for the movement in large water bodies, the laws to which the movement of land water obeys take the form of a partial differential equation of the heat conduction type. For this case it is well known that the solution of Cauchy's

* For example; estimation of the dislocation in the earth crust from the seismogrms, the estimation of the origin of air and water polution, etc.

problem is unique in the domain $t \geqq 0$, but not unique in $t<0$, when the initial condition is given at $t=0$.[1] Therefore the input of many hydrological system can be found only by approximation. The same difficulty occurs in the (A) type problems when the value of the input is essentially non-negative.[5]

The two kinds of the inverse transform, (A) and (B) are mathematically the same and both corresponds to solving the equation (1) as a Volterra's integral equation of the first kind. Many methods of solving the integral equations described in the textbooks of mathematics are not applicable because of the ununiqueness of the solution, and because the known functions are non-analytic ones including the observational errors. Therefore one had to devise the optimal method for each problem. When the time series of $f(t)$ and $g(t)$ are much longer than that of $\phi(t)$, it is advantageous to analyze the input-output relation indirectly through the statistical methods[2,3] rather than to solve the integral equation directly. This paper will not deal with this special case.

In most cases of hydrological problems, the observational data can be arranged so that the value of $f(t)$ in the time intervals $t<0$ and $t>T$ has practically no influence upon the value of $g(t)$ in the time interval $0 \leqq t \leqq T$, and the equation (1) can be approximated by,

$$g(t)=\int_0^t \phi(t-\tau)f(\tau)d\tau, \tag{2}$$

in the interval $0 \leqq t \leqq T$. The equation (2) is the same with,

$$g(t)=\int_0^t f(t-\tau)\phi(\tau)d\tau. \tag{3}$$

This paper will compare the methods of the inverse transform, i.e. to solve (2) or (3) for $f(t)$ or $\phi(t)$, which have been proposed for the problems in hydrology and the related fields, to find generally applicable methods as the tools of the processing of hydrological data. Nextly a result of the data processing of a tracer experiment using a method of the inverse transform is shown to illustrate its applicability.

2. Methods

In this section the methods of solving the equation (2) where $f(t)$ is unknown are treated. The equation (3) where $\phi(t)$ is unknown can be solved with the same methods. As the hydrological data are usually given as the time series with a small and constant time increment, the integral equation (2) can be replaced with a series using the quadrature formula, thus,

$$g_j=\sum_{i=1}^{j} \phi_{j-i+1} f_i, \qquad (i=1, 2, \cdots, n; j=1, 2, \cdots, n) \tag{4}$$

where the time interval $0 \leqq t \leqq T$ is divided into n sub-intervals, and each value of the functions is given as the average in each sub-interval. The equation (4) is rewritten in a matrix form,

$$g=Af, \tag{5}$$

where,

$$A=\begin{pmatrix} \phi_1 & 0 & 0 \cdots\cdots\cdots & 0 \\ \phi_2 & \phi_1 & 0 \cdots\cdots\cdots & 0 \\ \phi_3 & \phi_2 & \phi_1 \cdots\cdots\cdots & 0 \\ \cdots & \cdots & \cdots\cdots\cdots & \cdots \\ \phi_n & \phi_{n-1} & \phi_{n-2} \cdots\cdots\cdots & \phi_1 \end{pmatrix} \tag{6}$$

The equation (4) and (5) are simultaneous equations of the first degree concerning f_i. However, some modifications are needed to avoid the instability inherent in the original equation.

2.1. The least-square method

As the instability in the equation (2) is oscillatory[5], it can be reduced by a certain methods of smoothing. The least-square method is the most widely used method for this purpose. It can be applicable when it is previously assumed that $f_i=0$ for $i=m+1$, $m+2$, ..., n. From the condition,

$$\sum_{j=1}^{n}\{g_j-\sum_{i=1}^{m}(\phi_{j-i+1}f_i)\}^2=\text{min.}, \tag{7}$$

the normal equations become,

$$\sum_{j=1}^{n}(\phi_{j-k+1}g_j)-\sum_{j=1}^{n}\{\phi_{j-k+1}\sum_{i=1}^{m}(\phi_{j-i+1}f_i)\}=0. \tag{8}$$

The equation (8) is rewritten in a matrix form as,

$$({}^tA')g=({}^tA')A'f', \tag{9}$$

where A' is the small matrix consisting of the first m rows of the matrix A, and f' the one consisting of the first m lines of f. If the function $g(t)$ involves the unknown background[4], (9) should be replaced with,

$$({}^tA')(g-\bar{g})=\{({}^tA')-\bar{\phi}\}A'f', \tag{10}$$

where $\bar{g}$ and $\bar{\phi}$ are the matrices, the elements of which are the averages of g_i and ϕ_i for $i=1, 2, \ldots, n$, respectively. The equations (9) and (10) are the simultaneous equations of the first degree concerning f_i, and can be solved with a digital computer.

2.2 The method of Kuchment

In this method[5], the condition,

$$\sum_{i=1}^{n}f_i^2=R^2=\text{const.}, \tag{11}$$

is added to (4) to eliminate the instability in the solution of f_i. The solution that satisfies the condition (11) and also minimizes the residual error in the equation (4) is decided using the Lagrange's method of conditional extremum, i.e.,

$$\sum_{j=1}^{n}(g_j-\sum_{i=1}^{j}\phi_{j-i+1}f_i)^2+\lambda(\sum_{i=1}^{n}f_i{}^2-R^2)=\min., \tag{12}$$

where λ is the Lagrange's indeterminate coefficient. The equation (12) is reduced to,

$$\{({}^tA)A+\lambda\}f=({}^tA)g \tag{13}$$

The value of R in (11) must be decided according to the problem to be solved. Kuchment[5] has proposed to replace (11) with the following for the problems in which f_i must be non-negative*.

$$\sum_{i=1}^{n}|f_i|=W, \tag{14}$$

where,

$$W=\int_0^\infty f(t)dt \tag{15}$$

is calculated through the condition of the conservation of the mass, the energy, etc.** Actually the equation (13), a simulataneous equation of the first degree concerning f_i, is solved iteratively using different values of λ. The iteration is ended when the solution satisfies (14). A few iterations are, however, sufficient because the value of the left side of (14) changes monotonously with the value of λ. The condition (14) makes the value of $f(t)$ smaller than the proper value if some of the values of f_i become negative. Therefore, the solution f_i gotten by this method should be multiplied with a constant so as to satisfy (15).

2.3. The Fourier transform method

It is widely known that the Duhamel integral (2) is reduced to the product of f and ϕ in the Fourier domain as,

$$G=\Phi\cdot F, \tag{16}$$

where the capital letters mean the Fourier transforms of the functinos $g(t)$, $\phi(t)$, and $f(t)$, respectively. It is easy to find f by the inverse Fourier transform using F calculated with (16). The Fourier and the inverse Fourier transforms can be carried out numerically with a digital computer[6]. This method has frequently used in the field of the oscillation analysis and the related fields, but seldom in the field of hydrology probably because of the inherent instability. The author, according to the examination of the instability by Kuchment[5], proposes to apply a low-

* If the problem is to find ϕ, ϕ is known to be non-negative when the process in the system obeys hte heat-conduction type laws.

** Kuchment has described the case in which W equals theoretically to unity[5].

pass filter,

$$F(\omega)\rightarrow F(\omega)\exp\left(-\frac{\omega^2}{2\omega_0^2}\right), \tag{17}$$

to the Fourier spectrum F, where ω is the angular frequency, and ω_0 the cut-off angular frequency. The value of ω_0 must be determined according to the examination of the spectrum $F(\omega)$ so that the component of $F(\omega)$ with physical meaning are conserved and the component of instability are cut-off after the filtering.

2.4. Other methods

The unit graph method[7] and the moment method[8] are the popular methods of solving (3) for hydrologists. From the theory of unit graph, the output for a "unit input" is equal to the function of influence called unit graph. This method can be applied to the problem to find $\phi(t)$ only when the unit input is available. The moment method stands on the moment theorem which make it possible to calculate the moment of n-th order of the function of influence $\phi(t)$,

$$M_\phi^{(n)}=\int_0^\infty(-1)^n t^n\phi(t)e^{-st}dt, \tag{18}$$

from the moments of up to n-th order of $f(t)$ and $g(t)$. It is applicable when the function of influence $\phi(t)$ is to be found and when both the functional form of $\phi(t)$ and the relation between the parameters of $\phi(t)$ and the moments are known. As these two methods are not the universal ones, they shall not further treated in this paper.

3. Comparison of the methods

The methods described above are compared through their application to the two field data, a data in typically linear processes, and the one in non linear processes.

3.1. The routing of tracer in surface flow

In this section the problem is to find the concentration of a tracer $f(t)$ at a upper reach of a channel flow knowing the one $g(t)$ at a lower reach, the function of influence $\phi(t)$ being known from another experiment. Since the concentration of the tracer obeys the law of linear dispersion, it is assumed that the relation between $f(t)$ and $g(t)$ is linear. The known functions $\phi(t)$ and $g(t)$ are shown by the curve 5 in Fig. 1 and by the curve 4 in Fig. 2, respectively. The data is given with the time step of 0.5 minutes from $t=0$ to $t=16$ minutes. The solution by different methods are shown by the curves 1–4 in Fig. 1.

The solution obtained by the least-square method (curve 4) involves a significant oscillatory component with a short period. In this solution it has been assumed that $f(t)=0$ for $t\geqq 7.5$ min. The one with the method of Kuchment (curve 1) is smooth. The one by the Fourier transform method without filter (curve 3) involves a significant oscillation with a period of one minute, which is almost fully eliminated

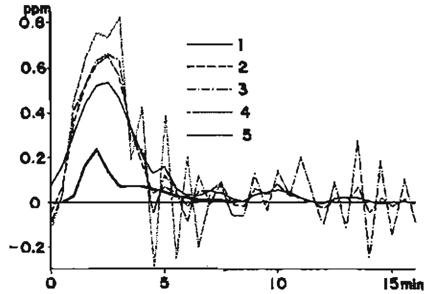

Fig. 1. The estimation of the concentration of the tracer at the upper reach by different methods. 1 — the method of Kuchment, 2 — the Fourier transform method (with filter), 3 — the Fourier transform method (without filter), 4 — the least square method, 5 — the function of influence given as the data in an arbitrary unit. The scale corresponds to the curves 1–4.

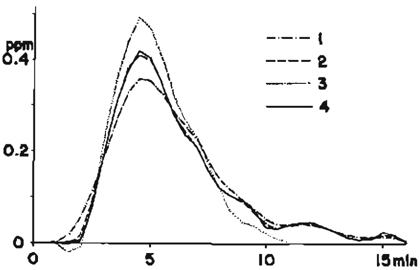

Fig. 2. The output functions reproduced from the different solutions of the input functions in Fig. 1. 1 — the method of Kuchment, 2 — the Fourier transform method, 3 — the leastsquare method, 4 — the original output function.

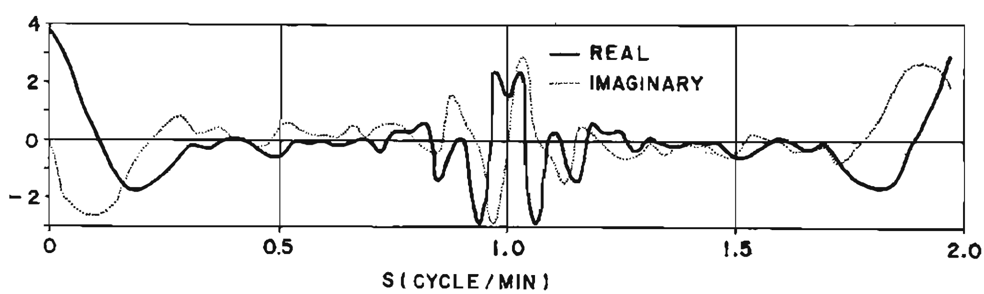

Fig. 3. The Fourier spectrum $F(\omega)$ of the curve 3 in Fig. 1 in an arbitrary unit.

by the filter ($\omega_0=0.5\ \mathrm{min}^{-1}$) as shown by the curve 2. The filter was designed so as to eliminate the conponent of the instability oscillation which seems to exist around the Nyquist frequency in the function $F(\omega)$ (Fig .3).

As the true value of $f(t)$ was not directly measured, the verification must be made by an indirect method. The values of $g(t)$ were reprocuced with (4) using the above solutions (Fig. 2). It can be seen that the curve of $g(t)$ reproduced using the solution by the Fourier transform method (curve 2) is the nearest to the original one (curve 4). The one reproduced using the solution by the method of Kuchment (curve 1) is lower at the peak and higher in the rising part than the original one. The one reproduced using the solution by the least-square method (curve 3) is, on the contrary, higher at the peak and lower elsewhere than the original one, taking a negative value near $t=1.5$ min.

3.2. The response of ground water level to rainfall

In this example, the problem is to find the function of influence $\phi(t)$ knowing the rainfall as the input function (curve 5 in Fig. 4) and the rise of the groundwater level as the output function (curve 4 in Fig. 4) of a groundwater system. As it has been

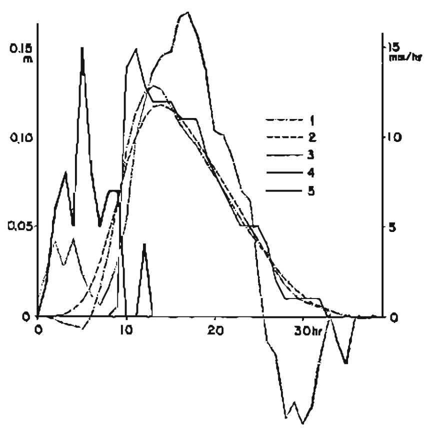

Fig. 4. The intensity of rainfall (curve 5), the rise of the groundwater level (curve 4), and the reproduction of the latter using the solutions by different methods (1 — the method of Kuchment, 2 — the Fourier transform method, 3 — the least-square method).

known that the rise of the groundwater level occurs only when the rainfall has exceeded a critical value (about 50 mm), the relationship between the rainfall and the rise of the groundwater level must be nonlinear. The linear relationshop (3) is however applied to this case for the test. The solution by different method are shown in Fig. 5. The solution by the least-square method (curve 3,) in which $\phi(t)$ is assumed to be zero for $t \geqq 26$ hr, oscillates extremely extending both in positive and negative domains. It is difficult to find a physical meaning from this curve. The solution by the method of Kuchment (curve 1) is of a reasonable form except for the negative value near $t=0$. The change of the solution due to the change of λ in the method of Kuchment (Fig. 6) shows that a smaller value of λ gives a oscillatory curve of $\phi(t)$ and a larger value of λ gives a too tame curve, the optimal value being 283.5. The

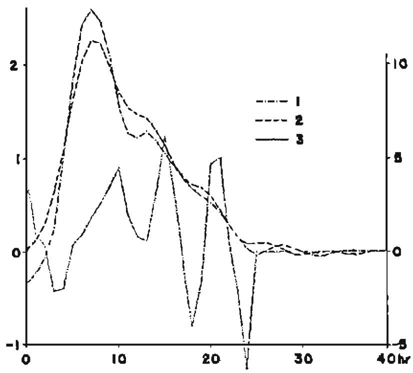

Fig. 5. The solution of the function of influence for the data shown in Fig. 4 by different methods (1 — the method of Kuchment, 2 — the Fourier transform method, 3 — the least-square method). The left scale corresponds to the curves 1 and 2, and the right one to the curve 3. The unit is arbitrary, but common to three curves.

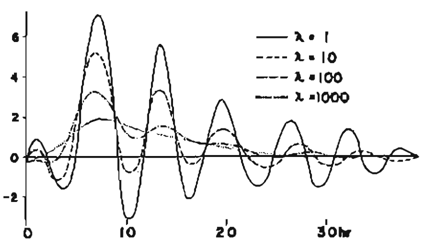

Fig. 6. The change of the solution of $\phi(t)$ due to the change of λ in the method of Kuchment (in an arbitrary unit).

solution by the Fourier transform method with a filter ($\omega_0 = 7$ hr) is shown by curve 2 in Fig. 5. The value of ω_0 was determined according to the fact that the solution without filter (Fig. 7) involved a large oscillatory component with a period of about 7 hours, which looked physically meaningless.

As the value of $\phi(t)$ can not be given by direct methods, the solutions were checked by the same method as in the previous section. The reproduced values using the solution by the method of Kuchment and the Fourier transform method are shown by the curves 1 and 2 in Fig. 4, respectively. General form of $g(t)$ is better reproduced when the solution by the method of Kuchment is used than when the solution by the Fourier transform method is used. The solution by the least-square method leads to a considerably poor reproduction of $g(t)$ as shown by curve 3 in Fig. 4. The difference between the reproduced and the original curves of $g(t)$ commonly seen at the rising part (see Fig. 4) is estimated to be due to the nonlinearity of the recharge process in the groundwater system. Inversely, it is possible to say that a significant discrepancy between the original and the reproduced values indicates the nonlinearity involved in the process.

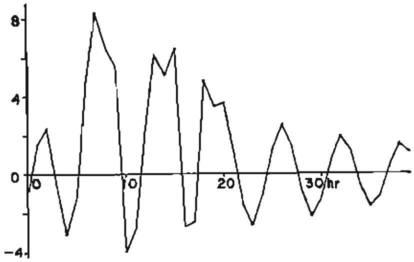

Fig. 7. The solution of $\phi(t)$ by the Fourier transform method when no filter is applied (in an arbitrary unit).

As a conclusion of this chapter, it is pointed out that the least-square method is not suitable for such problems as these examples. The method of Kuchment and the Fourier transform method are considered to give satisfactory results. Further discussion will be made in Chapter 5.

4. Application to the tracer experiment

Hitherto the inverse transform of the Duhamel integral has been carried out usually by the least-square method. As the components of physical meaning in the solution is burried in the oscillatory component when this method is applied, the analysis of the function $\phi(t)$ or $f(t)$ gotten as the solution has been limited to a preliminary one. As it is recognized through the previous chapter that the solution can be gotten with a considerable accuracy using a suitable method, it seems practical to use the inverse transform of the Duhamel integral as a universal method of the data processing for deducing the quantities representing the characteristics of any hydrologic system from the observational data.

In this chapter an attempt at the data processing of the result of the tracer experiment is described as an example. The tracing of the overland flow was carried out in a model drainage basin on a reduced scale shown in Fig. 8. The valley (2–3) to which the surface runoff inflows from the slopes on the both sides was drained to the channel (3–4). The artificial rainfall of 100 mm/hr was maintained so that the flow was steady throughout the drainage basin during the tracer experiment. The tracer (one gram of uranine) was added to the surface flow at the points 3, 1C, 1A, and 1B in the four runs, respectively, and detected always at the V-notch station (4) at the end of the channel. As the coefficient of runoff was nearly 100%, it can be assumed that all of the tracer passed the detection point. The concentration of the tracer at the detection point in the four runs is shown in Fig. 9. In the first run, the tracer was injected at the point 3 and detected at the point 4.

The concentration of the tracer observed is expressed as,

$$q_{w4}C_4(t)=M\phi_{34}(t), \tag{19}$$

where q_w is the water discharge, C the concentration of the tracer, M the quantity of the tracer injected, and ϕ the function of influence for the reach. The other

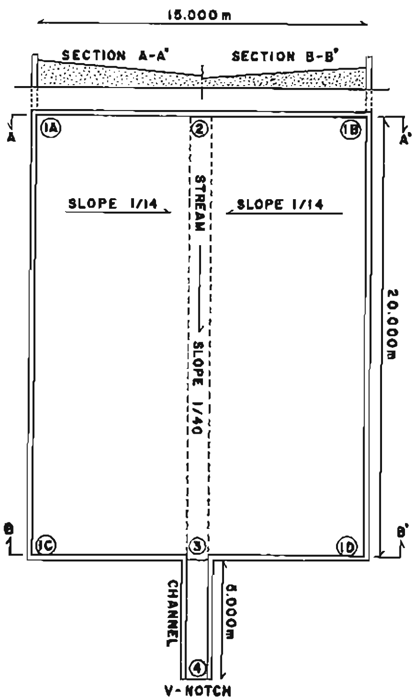

Fig. 8. A sketch of the model drainage basin where the tracing of the overland flow was carried out.

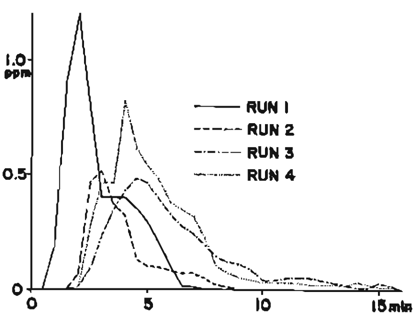

Fig. 9. The time changes of C_4, the concentration of the tracer at the detection point 4.

subscript represent the location. The quantity ϕ_{34} is easily calculated from (19) (see the bottom part of Fig. 10), using the value of q_{w4} calculated with the equation of the mass conservation,

$$\int_0^\infty q_{w4}C_4(t)dt=M, \tag{20}$$

In the second run, the tracing was carried out through the route 1C–3–4 to investigate the characteristics of the overland flow on the slope surface. As the relations,

$$q_{w4}C_4(t)=q_{w3}\int_0^t \phi_{34}(t-\tau)C_3(\tau)d\tau, \tag{21}$$

$$q_{w4}=q_{w3}, \tag{22}$$

hold for the reach 3–4, the concentration of the tracer at the point 3, C_3, was calculated through the inverse transform of the Duhamel integral (21). Then using the relation,

$$q_{w3}C_3(t)=M\phi_{13}(t), \tag{23}$$

for the process on the slope surface 1–3, ϕ_{13} was calculated as shown at the top of Fig. 10. In the third and fourth runs, the tracing was carried out through the routes

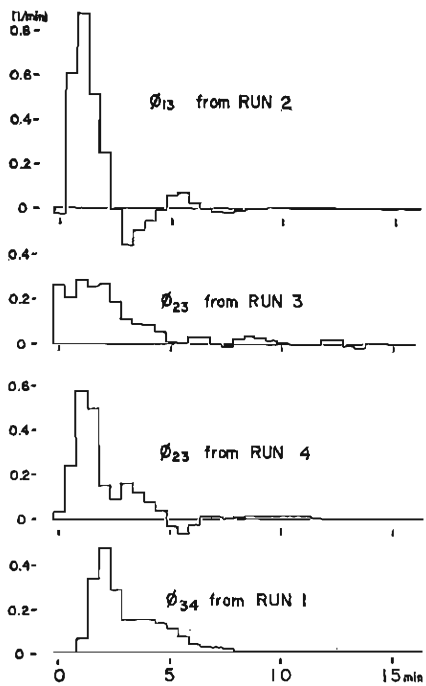

Fig. 10. The values of the function of flunience for different reaches.

1A–2–3–4, and 1B–2–3–4, respectively. The quantity $q_3=q_{w3}C_3$ was calculated in the same way as in the second run. On the other hand, the quantity $q_2=q_{w2}C_2$ was calculated using the relation for the process on the slope surfaces 1A–2 and 1B–2,

$$q_2(t)=M\phi_{12}(t), \tag{24}$$

and assuming,

$$\phi_{12}(t)=\phi_{13}(t).$$

Then, the quantity ϕ_{23} was calculated through the inverse transform of the equation,

$$q_3(t)=\int_0^\infty \phi_{23}(\tau)q_2(t-\tau)d\tau, \tag{25}$$

which holds for the process along the valley 2–3. The results are shown in the center of Fig. 10. The inverse transform was carried out always by the method of Kuchment.

The curves of the function ϕ for individual reaches shown in Fig. 10 involve some unnaturalness. Some of them show negative values at certain intervals of t. The curves of ϕ_{23} from the two experiments differ considerably from each other. These unnatural occurrences can be explained by the fact that a large scale eddy was found in the channel 3–4 being coloured with the tracer. The occurrence of the negative value in ϕ is attributable to the fluctuation of the eddy, or directly, to the lack of the reproducibility in ϕ_{34}. The difference in the two curves of ϕ_{23} is atributable to the facts that the clouds of the tracer which arrived at the point 2 from the slope surfaces of different sides flowed along the valley 2–3 on different sides, and that these clouds were then entrained into the eddy in the channel 3–4 in different ways to make the different patterns of the time change of C_4 (c.f. the curves of runs 3 and 4 in Fig. 9). The two curves of C_3 calculated from the data of runs 3 and 4 are compared in Fig. 11.

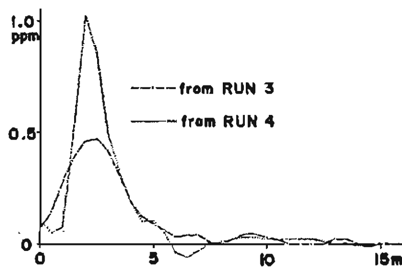

Fig. 11. Difference in the estimation of $C_3(t)$ between in the third and fourth runs.

The one dimensional dispersion equation is written as,

$$\frac{\partial C}{\partial t}=\frac{\partial}{\partial x}\left(D\frac{\partial C}{\partial x}\right)-u\frac{\partial C}{\partial x}, \tag{27}$$

where C is the concentration of the tracer, D the coefficient of dispersion, u the mean velocity, x the distance along the flow, and t the time. When the flow is uniform (both u and D are independent of x), the solution of (27) for impulsive input is[9],

$$C=\frac{Mu}{2\sqrt{\pi Dt}\,q}\exp\left\{-\frac{(x-ut)^2}{4Dt}\right\}, \tag{28}$$

where M is the quantity of the tracer injected, and q is the discharge. As C is expressed as,

$$C=M\phi/q, \tag{29}$$

ϕ becomes,

$$\phi(t)=\frac{u}{2\sqrt{\pi Dt}}\exp\left\{-\frac{(x-ut)^2}{4Dt,}\right\}, \tag{30}$$

or,

$$\ln(\sqrt{t}\,\phi)=\frac{ux}{2D}-\frac{1}{4D}\left(\frac{x^2}{t}+u^2t\right)+\ln\left(\frac{u}{2\sqrt{\pi D}}\right). \tag{31}$$

The expression (31) means that $\ln(\sqrt{t}\,\phi)$ becomes maximum at $t=x/u$ and that $\ln(\sqrt{t}\,\phi)$ and $(x^2/t+u^2t)$ are in a linear relationship with the coefficient $-1/4D$ when t is the independent variable. Though the flow in the model drainage basin is not uniform, the above formulae were applied to the flow at the individual reach as a first approximation. Figs. 12 and 13 are drawn in order to determine the values of u, and then those of D according to (31). The results are shown in Tables 1 and 2 with the ones obtained by other methods.

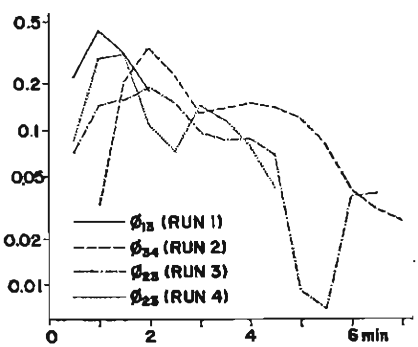

Fig. 12. The plot of $\sqrt{t}\,\phi$ versus t for the determination of the value of u at different reaches according to (31).

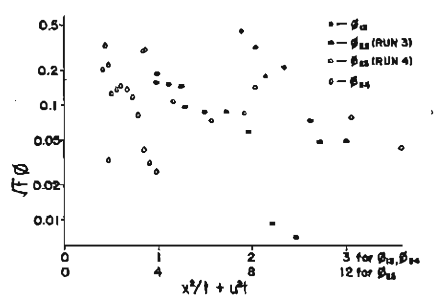

Fig. 13. Determination of the values of D at different reaches according to (31). The units are taken arbitrarily.

Table 1. The numerical data obtained from the tracer experiments

Reaches	u (cm/sec)	D (cm^2/sec)
IC–3	$u_{13}=12.5$	$D_{13}=1{,}220$
2–3	$u_{23}=19.6/27.8$	$D_{23}=7{,}470/10{,}300$
3–4	$u_{34}=\ 4.9$	$D_{34}=\ 452$

Table 2. The discharge at the point 4 (q_{w4}) by different methods.

methods	q_{w4} (m^3/min)
V-notch data	0.37
rainfall int. × area	0.50
u_{34} × cross sectional area	0.76
Run 1 with (20)	0.37
Run 2 with (20)	0.82
Run 3 with (20)	0.53
Run 4 with (20)	0.47

Among the numerical values of q_{w4} in Table 1, the one by the V-notch data is the most reliable. The value according to the mean velocity u_{34} is too large. The cause is supposed to be that the eddy in the channel (3–4) carried a part of the tracer-cloud rapidly and made a peak in the curve of $C_4(t)$ too early. The variation among the values of q_{w4} by the dilution method using (20) is thought to be due to the error in the integration of $C_4(t)$ because the time interval of the succeeding measurements of C_4 was too long compared with the rate of its time change (see Fig. 9). A small quantity of the infiltration on the slope surface made the quantity of the tracer at the detection point somewhat smaller than at the injection point. It may be the reason why the values of q_{w4} are too large according to runs 2, 3, and 4.

5. Discussion and conclusions

Though it is impossible to decide rigorously which of the solutions by different methods is the best when its true value is unknown, the result of Chapter 3 suggests the existence of some criteria for good solutions. The one is the difference between the reproduced and the original values of $g(t)$, which may be called the criterion of the "precision" of the solution. The other is the degree of the instability oscillation involved in a solution, which may be called the criterion of the "reasonableness".

Among the methods compared in Chapter 3 the least-square method could give no satisfactory solutions. A probable cause is that an indirect (therefore ineffective) procedure for the elimination of the instability is adopted in this method. It can also be agreed that the two examples were not fitted to this method. As this method requires that the number of the data be much larger than that of the unknowns, the application of it may be effective when the function of influence is to be found and the time bases of the input and output functions given as the data are much longer than that of the function of influence. In such a case, the result is inevitably an average of the functions of influence for some succeeding sets of input and output. As the characteristics of a hydrologic system are usually instantaneous one, it is desirable to find them for each set of input and output. Therefore the practicability of the least-square method is limited only to some special cases.

The difference between the method of Kuchment and the Fourier transform method in their "precision" and "resonability" was not obvious in Chapter 3. It can be, however, said that the method of Kuchment always gives a fairly good solution. On the other hand, the result by the Fourier transform method seems to depend on the design of the filter. In the first example the component of the instability oscillation was found near the Nyquist frequency (1 cycle/min.) in the Fourier spectrum $F(\omega)$ (see Fig. 3). It was easy to design a filter which eliminates only the instability component. In the second example, the Fourier transform of the input function $f(t)$ had a near-zero value at the period of 7 hours, which made a peak at the same period in the Fourier spectrum of $\phi(t)$. It is probable that the filter which was designed to eliminate the oscillation with the period of 7 hours clipped a part of physically significant components from the solution. Actually the comparison between the reproduced and the original values of $g(t)$ (Fig. 4) suggests that the solution by the Fourier transform method have been over-filtered. It is concluded that the Fourier transform method can give the best solusion if the filter is well designed according to the examination of the Fourier spectra $F(\omega)$, $\Phi(\omega)$, and $G(\omega)$, and that the method of Kuchment is the best when the procedure of the inverse transform is required to be fixed (e.g. for automatic data processing).

If the inverse transform of Duhamel integral were not applied, the data of the tracer experiment would not give reasonable values of the velocity and the dispersion coefficient of the flow at individual reaches, but only a rough estimate of them averaged from the injection points to the detection point of the tracer. Though the values of the velocity and the dispersion coefficient shown in Chapter 4 involve a considerable error, it may be expected that the error is reduced to within 10% by the improvements of the exprimental procedure and of the measurement. It can be expected that the application of new methods of the inverse transform of the Duhamel integral to the processing of any hydrologic data offers a more quantitative and more effective basis for the analysis of the hydrogic processes than the conventional methods.

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